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# Obtaining the Weyl tensor from the Bel-Robinson tensor

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**Abstract** The algebraic study of the Bel-Robinson tensor proposed and initiated in a previous work (Gen. Relativ. Gravit. **41**, see ref [11]) is achieved. The canonical form of the different algebraic types is obtained in terms of Bel-Robinson eigen-tensors. An algorithmic determination of the Weyl tensor from the Bel-Robinson tensor is presented.

**Keywords** Bel-Robinson tensor · Gravitational superenergy · Petrov-Bel classification

## 1 Introduction

The *super-energy Bel tensor* is a rank 4 tensor, quadratic in the Riemann tensor, which plays an analogous role for gravitational field to that played by the energy tensor for electromagnetism. It was introduced by Bel to define intrinsic states of gravitational radiation [1] [2] [3]. In the vacuum case, the Bel tensor coincides with the *super-energy Bel-Robinson tensor*  $T$ , built with the same expression by replacing the Riemann tensor with the Weyl tensor  $W$ , namely:

$$T_{\alpha\mu\beta\nu} = \frac{1}{4} (W_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma} W_{\mu\rho\nu\sigma} + *W_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma} *W_{\mu\rho\nu\sigma}) , \quad (1)$$

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where  $*$  denotes the Hodge dual operator.

In recent years a lot of work has been devoted to analyzing the properties of the super-energy tensors, and to studying their generalization to any dimension and to any physical field [4]. In a recent paper [5], where the dynamic laws of super-energy are accurately analyzed, up-to-date references on the Bel and Bel-Robinson tensors can be found.

Some algebraic properties of the Bel-Robinson tensor (BR tensor) were studied by Debever early on [6] [7]. The intrinsic algebraic characterization of a BR tensor was obtained a few years ago in a paper [8] that presents the conditions on the BR tensor playing a similar role to that played by the algebraic Rainich [9] conditions for the electromagnetic field.

For a rank 2 tensor satisfying the algebraic Rainich conditions an exhaustive algebraic study is fully known (see the original work by Rainich [9] or the more recent one [10]). Nevertheless, the algebraic study of the BR tensor has not been wholly achieved. In a recent work [11] we have posed all the different aspects of a full algebraic study of the BR tensor:

(I) Algebraic classification of a BR tensor  $T$ : the symmetries of the BR tensor  $T$  allow us to consider and analyze it as a linear map on the nine dimensional space of the traceless symmetric tensors. What algebraic classification follows on from this study? What relationship exists between these BR classes and the Petrov-Bel types of the Weyl tensor?

(II) Canonical form of the BR tensor  $T$  in terms of its invariant spaces and scalars: for every algebraic type of the BR tensor, the eigenvalues and eigenvectors should be analyzed, as well as the canonical expression of  $T$  in terms of them. What relationship exists between the spaces and the scalar invariants of both the BR tensor and the Weyl tensor?

(III) Expression of the Weyl tensor in terms of the BR tensor: it is known that the BR tensor  $T$  determines the Weyl tensor  $W$  up to a duality rotation. But the explicit expression of  $W$  in terms of  $T$  has not been established.

In [11] we have tackled the algebraic problems of the BR tensor stated in the three points above: we have solved the first one and have given preliminary results on points (II) and (III). The goal of the present work is to fully solve them and to gain a comprehensive algebraic understanding of the BR tensor.

In section 2 we summarize some results on point (I) obtained in [11]: we introduce some notation and present the classification of the non vanishing BR tensors in nine classes that can be distinguished considering both, the eigenvalue multiplicity and the degree of the minimal polynomial. The correspondence of these classes with the Petrov-Bel types and some type I degenerations is also outlined.

Section 3 is devoted to studying the eigentensors and invariant spaces of the BR tensor and to analyze the their relationship with the Weyl canonical bivectors. The canonical form of a BR tensor in terms of its eigentensors is also presented.

In section 4 we study the Segrè types of the BR tensor and offer them in the form of a flow chart that explains the relationship between the different BR types.

In section 5 we give the algorithms that explicitly determine (up to a duality rotation) the Weyl tensor in terms of the BR tensor.

Some notation used in the paper is presented in appendix A. In appendix B we summarize basic properties of the frames of vectors, bivectors and symmetric tensors. Appendix C offers some technical results that are required in section 5.

## 2 Algebraic types of the Bel-Robinson tensor

We shall note  $g$  the space-time metric with signature  $\{-, +, +, +\}$ . The conventions of signs are those in the book by Stephani *et al.* [12]. We take for the Bel-Robinson tensor (BR tensor)  $T$  the expression (1), that differs in a factor from the original one by Bel [3].

In [11] we have shown that the algebraic types of the BR tensor correspond to a refinement of the Petrov-Bel classification. In order to explain this correspondence, we summarize the algebraic classification of the Weyl tensor in next subsection.

### 2.1 Invariant classification of the Weyl tensor

A self-dual 2-form is a complex 2-form  $\mathcal{F}$  such that  $*\mathcal{F} = i\mathcal{F}$ . We can associate biunivocally with every real 2-form  $F$  the self-dual 2-form  $\mathcal{F} = \frac{1}{\sqrt{2}}(F - i * F)$ . In short, here we refer to a self-dual 2-form as a *SD bivector*. The endowed metric on the 3-dimensional complex space of the SD bivectors is  $\mathcal{G} = \frac{1}{2}(G - i \eta)$ ,  $\eta$  being the metric volume element of the space-time, and  $G$  being the metric on the space of 2-forms,  $G = \frac{1}{2}g \wedge g$ . Here  $\wedge$  denotes the double-forms exterior product (see Appendix A.1.5).

The algebraic classification of the Weyl tensor  $W$  can be obtained [13], [3] by studying the traceless linear map defined by the self-dual (SD) Weyl tensor  $\mathcal{W} = \frac{1}{2}(W - i * W)$  on the SD bivectors space. This *SD-endomorphism* (see notation in Appendix A.2) has associated the complex scalar invariants  $a = \text{Tr } \mathcal{W}^2$  and  $b = \text{Tr } \mathcal{W}^3$ , and its characteristic equation is of degree three.

Then, the Petrov-Bel classification follows taking into account both the eigenvalue multiplicity and the degree of the minimal polynomial. In the algebraically general case (type I) the characteristic equation admits three different roots and occurs when  $6b^2 \neq a^3$ . When  $6b^2 = a^3 \neq 0$  there is a double root and a simple one, and therefore the minimal polynomial distinguishes between types D and II. Finally, if  $a = b = 0$  all the roots are equal, and so zero, and the Weyl tensor is type O, N or III, depending on the minimal polynomial.

On the other hand, some classes of algebraically general space-times have been considered in literature [14]. This refinement of the Petrov-Bel classification is based on the Weyl scalar invariant  $M = \frac{a^3}{b^2} - 6$ . The complex invariant  $M$  does not vanish for type I Weyl tensors. When  $M$  is a positive real or infinity, the Weyl tensor is called of type  $\text{IM}^+$  or  $\text{IM}^\infty$ , respectively. These classes have the following geometric interpretation [14] [15]:  $M$  is a

positive real or infinity if, and only if, the four Debever null directions span a 3-plane. When  $M$  is a negative real we have the class  $IM^-$ , which has been interpreted in terms of permutability properties of the null Debever directions [16]. We denote  $I_r$  the class of non ‘degenerate’ type I Weyl tensors, that is to say, those with non real invariant  $M$ .

A detailed analysis of these ‘degenerate’ algebraically general classes can be found in [16], where they are also interpreted as the space-times where the electric and magnetic parts of the Weyl tensor with respect to a (non necessarily time-like) direction are aligned. These classes also contain the purely electric and purely magnetic space-times which have been accurately studied in [17].

## 2.2 Invariant classification of the BR tensor

The expression (1) of the BR tensor may be written in terms of the SD Weyl tensor as:

$$T_{\alpha\mu\beta\nu} = \mathcal{W}_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma} \overline{\mathcal{W}}_{\mu\rho\nu\sigma}, \quad (2)$$

where  $\bar{\phantom{x}}$  stands for complex conjugate.

The BR tensor  $T$  is a fully symmetric traceless tensor and, consequently, it defines a traceless symmetric endomorphism on the 9-dimensional space of the traceless symmetric tensors [11]. Expression (2) implies that the properties of  $T$  as an endomorphism follow from the properties of  $\mathcal{W}$  as an endomorphism and, as a consequence, the classes of  $T$  are associated with the classes of  $\mathcal{W}$  presented in subsection above. A first significant fact is the relationship between the respective eigenvalues [11]:

**Lemma 1** *The BR tensor  $T$  defines an endomorphism of the 9-dimensional space of the traceless symmetric tensors that admits, generically, three real eigenvalues  $t_i$ , and three pairs of complex conjugates eigenvalues  $\tau_i, \bar{\tau}_i$ . In terms of the Weyl eigenvalues  $\rho_i$  we have:  $t_i = |\rho_i|^2$ , and  $\tau_i = \rho_j \bar{\rho}_k$  for  $(i, j, k)$  a pair permutation of  $(1, 2, 3)$ .*

On the other hand, the analysis of the eigenvalue multiplicity leads to the following result [11]:

**Proposition 1** *Taking into account the eigenvalue multiplicity we can distinguish seven classes of non vanishing BR tensors which are related with the Weyl tensor types as follows:*

*Type  $I_r$  iff  $T$  has nine different eigenvalues, three real ones and three pairs of complex conjugate:  $\{t_1, t_2, t_3, \tau_1, \tau_2, \tau_3, \bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3\}$ .*

*Type  $IM^-$ ,  $M \neq -6$ , iff  $T$  has six different eigenvalues, a simple and a double real ones, and two double and two simple complex conjugate eigenvalues:  $\{t, t, t_3, \tau, \tau, \tau_3, \bar{\tau}, \bar{\tau}, \bar{\tau}_3\}$ .*

*Type  $IM^+$  iff  $T$  has six different real eigenvalues, three simple ones and three double ones:  $\{t_1, t_2, t_3, \tau_1, \tau_2, \tau_3, \tau_1, \tau_2, \tau_3\}$ .*

*Type  $IM^{[-6]}$  iff  $T$  has three triple eigenvalues, one real and a pair of complex conjugate:  $\{t, t, t, \tau, \tau, \bar{\tau}, \bar{\tau}, \bar{\tau}\}$ .*

Type  $IM^\infty$  iff  $T$  has three different real eigenvalues with multiplicities 2, 2, 5:  $\{t, t, 0, 0, 0, -t, 0, 0, -t\}$ .

Types  $D$  and  $II$  iff  $T$  has three real eigenvalues with multiplicities 1, 4, 4:  $\{4t, t, t, t, -2t, -2t, t, -2t, -2t\}$ .

Types  $N$  and  $III$  iff  $T$  has a sole vanishing eigenvalue with multiplicity 9:  $\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$ .

The above classification of the BR tensor may be refined by considering the degree of the minimal polynomial [11]:

**Proposition 2** *The Petrov-Bel types of the two last cases in proposition 1 can be distinguished by the degree of the minimal polynomial of the BR tensor:*

*Types  $D$  and  $II$  have a minimal polynomial of degree 3 and 6, respectively.*

*Types  $N$  and  $III$  have a minimal polynomial of degree 2 and 3, respectively.*

It is worth remarking that the nine types of BR tensor that we find in the above propositions have a definition in terms of invariant properties of the BR tensor (eigenvalues and minimal polynomial). Consequently, these BR types admit an intrinsic characterization with explicit expressions involving the sole BR tensor. These expressions and their presentation in an algorithmic form can be found in [11].

### 3 Eigentensors and canonical forms of the Bel-Robinson tensor

The algebraic study of the BR tensor presented in [11] and summarized in section above classifies  $T$  as an endomorphism. Nevertheless, a whole study of this endomorphism also requires the study of its eigenspaces and invariant spaces. Note that, in this case, an eigenvector of  $T$  is a traceless symmetric tensor (in short, eigentensor).

The analysis of the eigentensors (and other characteristic tensors in the invariant spaces) of the BR tensor  $T$  enables us to accomplish in this section point (II) stated in the introduction. Indeed, for every algebraic type, we can obtain the canonical expression of  $T$  in terms of the eigentensors and the eigenvalues. We also present, for every Bel-Robinson type, the relationship between the eigentensors of  $T$  and the eigenbivectors of the associated Weyl tensor.

#### 3.1 Type I

The Weyl tensor is Petrov type I when it admits three different eigenvalues  $\{\rho_i\}$ . Then, an orthonormal frame  $\{\mathcal{U}_i\}$  of SD eigenbivectors exists and, consequently, the Weyl tensor takes the canonical expression [18]:

$$\mathcal{W} = - \sum_{i=1}^3 \rho_i \mathcal{U}_i \otimes \mathcal{U}_i \quad (3)$$

Thus, in this case we have three *canonical bivectors*  $\mathcal{U}_i$ , fixed up to sign, and six *canonical 2-planes* defined by the associated 2-forms  $\mathcal{U}_i$  and  $*\mathcal{U}_i$ .

From (2) and (3) we can compute the BR tensor. Then, taking into account lemma 1 and relations given in Appendix B, we obtain the following:

**Proposition 3** *The BR tensor associated with a type I Weyl tensor admits the canonical expression*

$$T = \sum_{i=1}^3 t_i \Pi_i \otimes \Pi_i + \sum_{(ijk)} \tau_i \Pi_{jk} \otimes \Pi_{jk} + \sum_{(ijk)} \bar{\tau}_i \Pi_{kj} \otimes \Pi_{kj} \quad (4)$$

where  $\Pi_i = \mathcal{U}_i \cdot \bar{\mathcal{U}}_i$  are eigentensors associated with the real eigenvalues  $t_i$  and  $\Pi_{jk} = \mathcal{U}_j \cdot \bar{\mathcal{U}}_k (= \bar{\Pi}_{kj})$  are eigentensors associated with the (generically) complex eigenvalues  $\tau_i$ ,  $(i, j, k)$  being a pair permutation of  $(1, 2, 3)$ .

Note that  $2\Pi_i = U_i^2 + *U_i^2$ . Thus  $\Pi_i$  are the structure tensors of the three 2+2 almost-product structures defined by the principal planes.

*Type  $I_r$*

In the most regular case the BR tensor has nine different eigenvalues (see proposition 1). Thus, we have:

**Proposition 4** *A type  $I_r$  BR tensor admits the canonical expression (4).*

*All the eigenvalues are different and  $\{\Pi_i, \Pi_{jk}, \Pi_{kj}\}$  is the (sole) normalized frame of eigentensors.*

*Type  $IM^+$*

In this case the BR tensor is like the regular case with  $\tau_i = \bar{\tau}_i$  (see proposition 1). Thus, we have:

**Proposition 5** *A type  $IM^+$  BR tensor admits the canonical expression*

$$T = \sum_{i=1}^3 t_i \Pi_i \otimes \Pi_i + \sum_{(ijk)} \tau_i (\Pi_{jk} \otimes \Pi_{jk} + \Pi_{kj} \otimes \Pi_{kj}) \quad (5)$$

*The eigenvalues  $t_i$  are simple and are associated with the eigentensors  $\Pi_i$ , and the (real) eigenvalues  $\tau_i$  are double and are associated with the eigenplanes  $S_i = \langle \Pi_{jk}, \Pi_{kj} \rangle$ ,  $(i, j, k)$  being a pair permutation of  $(1, 2, 3)$ .*

*Type  $IM^\infty$*

We can see this case as a degeneration of the previous one making  $t_1 = t_2 = \tau_3 = 0$  and  $t \equiv t_2 = t_3 = -\tau_1$  (see proposition 1). Thus, we have:

**Proposition 6** *A type  $IM^\infty$  BR tensor admits the canonical expression*

$$T = t (\Pi_2 \otimes \Pi_2 + \Pi_3 \otimes \Pi_3 - \Pi_{23} \otimes \Pi_{23} - \Pi_{32} \otimes \Pi_{32}) \quad (6)$$

*The eigenvalues  $\pm t$  are double and are associated with the eigenplanes  $S_+ = \langle \Pi_2, \Pi_3 \rangle$  and  $S_- = \langle \Pi_{23}, \Pi_{32} \rangle$ , respectively. The five-dimensional eigenspace  $S_0 = \langle \Pi_1, \Pi_{12}, \Pi_{21}, \Pi_{13}, \Pi_{31} \rangle$  is associated with a vanishing eigenvalue.*

Type  $IM^-$

In this case the BR tensor is like the regular case with  $t \equiv t_1 = t_2$  and  $\tau \equiv \tau_1 = \tau_2$  (see proposition 1). Thus, we have:

**Proposition 7** *A type  $IM^-$  BR tensor admits the canonical expression*

$$T = t(\Pi_1 \otimes \Pi_1 + \Pi_2 \otimes \Pi_2) + t_3 \Pi_3 \otimes \Pi_3 + \tau(\Pi_{23} \otimes \Pi_{23} + \Pi_{31} \otimes \Pi_{31}) \\ + \bar{\tau}(\Pi_{32} \otimes \Pi_{32} + \Pi_{13} \otimes \Pi_{13}) + \tau_3 \Pi_{12} \otimes \Pi_{12} + \bar{\tau}_3 \Pi_{21} \otimes \Pi_{21} \quad (7)$$

The eigenvalues  $t_3$ ,  $\tau_3$  and  $\bar{\tau}_3$  are simple and are associated with the eigentensors  $\Pi_3$ ,  $\Pi_{12}$  and  $\Pi_{21}$ , respectively. The eigenvalues  $t$ ,  $\tau$  and  $\bar{\tau}$  are double and are associated, respectively, with the eigenplanes  $S_t = \langle \Pi_1, \Pi_2 \rangle$ ,  $S_\tau = \langle \Pi_{23}, \Pi_{31} \rangle$  and  $S_{\bar{\tau}} = \langle \Pi_{32}, \Pi_{13} \rangle$ .

Type  $IM^{[-6]}$

We can see this case as a degeneration of the previous one making  $t_3 = t$  and  $\tau_3 = \tau$  (see proposition 1). Thus, we have:

**Proposition 8** *A type  $IM^{[-6]}$  BR tensor admits the canonical expression*

$$T = t(\Pi_1 \otimes \Pi_1 + \Pi_2 \otimes \Pi_2 + \Pi_3 \otimes \Pi_3) + \tau(\Pi_{12} \otimes \Pi_{12} + \Pi_{23} \otimes \Pi_{23} + \Pi_{31} \otimes \Pi_{31}) \\ + \bar{\tau}(\Pi_{21} \otimes \Pi_{21} + \Pi_{32} \otimes \Pi_{32} + \Pi_{13} \otimes \Pi_{13}) \quad (8)$$

Associated with the eigenvalues  $t$ ,  $\tau$  and  $\bar{\tau}$  we have, respectively, the three-dimensional eigenspaces  $P_t = \langle \Pi_1, \Pi_2, \Pi_3 \rangle$ ,  $P_\tau = \langle \Pi_{12}, \Pi_{23}, \Pi_{31} \rangle$  and  $P_{\bar{\tau}} = \langle \Pi_{21}, \Pi_{32}, \Pi_{13} \rangle$ .

### 3.2 Type II

A type II Weyl tensor has a simple eigenvalue  $-2\rho$  and a double one  $\rho$ , and it admits the canonical expression [18]:

$$\mathcal{W} = 2\rho \mathcal{U} \otimes \mathcal{U} - \rho \tilde{\mathcal{L}} \otimes \mathcal{K} + \rho \mathcal{L} \otimes \mathcal{L} \quad (9)$$

where  $\{\mathcal{U}, \mathcal{L}, \mathcal{K}\}$  is a (Jordan) null frame of SD bivectors.

The *canonical normalized bivector*  $\mathcal{U}$  is the eigenbivector associated with the simple eigenvalue. Associated with the double eigenvalue we have the invariant plane  $\langle \mathcal{L}, \mathcal{K} \rangle$ , the *canonical null bivector*  $\mathcal{L}$  being the unique eigendirection in this plane.

As a consequence of lemma 1, we have  $t \equiv t_2 = t_3 = \tau_1$ ,  $t_1 = 4t$  and  $\tau_2 = \tau_3 = -2t$ . Then, from (2) and (9), and taking into account relations given in Appendix B, we can compute the BR tensor and obtain the following:

**Proposition 9** *A BR tensor of type II admits the canonical expression*

$$T = 4t \Pi \otimes \Pi - 2t(\Omega \tilde{\otimes} N + \overline{\Omega} \tilde{\otimes} \overline{N} - N \otimes N - \overline{N} \tilde{\otimes} \overline{N}) \\ + t(\Lambda \tilde{\otimes} K + M \tilde{\otimes} \overline{M} - \Lambda \tilde{\otimes} \overline{M} - \Lambda \tilde{\otimes} M + \Lambda \tilde{\otimes} \Lambda) \quad (10)$$

where the (complex) null frame  $\{\Pi, \Lambda, K, N, \overline{N}, \Omega, \overline{\Omega}, M, \overline{M}\}$  is defined in terms of the Weyl canonical frame  $\{\mathcal{U}, \mathcal{L}, \mathcal{K}\}$  as in (66). The eigenvalue  $4t$  is simple and it is associated with the eigentensor  $\Pi$ . Associated with the quadruple eigenvalue  $t$  we have the eigentensor  $M_I$  and the invariant space  $Q_t = \{\Lambda, M_R, K\}$ , with  $\Lambda$  the sole eigentensor. Associated with the quadruple eigenvalue  $-2t$  we have the invariant spaces  $Q_R = \{N_R, \Omega_R\}$  with  $N_R$  the sole eigentensor, and  $Q_I = \{N_I, \Omega_I\}$  with  $N_I$  the sole eigentensor.

In the above proposition and in that follows,  $\Psi_R$  and  $\Psi_I$  stand, respectively, for the real and imaginary parts of a complex symmetric tensor  $\Psi$ .

### 3.3 Type D

A type D Weyl tensor also has a simple eigenvalue  $-2\rho$  and a double one  $\rho$ , and it admits the canonical expression [18]:

$$\mathcal{W} = 2\rho \mathcal{U} \otimes \mathcal{U} - \rho \mathcal{L} \tilde{\otimes} \mathcal{K} \quad (11)$$

where  $\{\mathcal{U}, \mathcal{L}, \mathcal{K}\}$  is a null frame of SD bivectors.

The *canonical unitary bivector*  $\mathcal{U}$  is the eigenbivector associated with the simple eigenvalue. Associated with the double eigenvalue we have the eigenplane  $\langle \mathcal{L}, \mathcal{K} \rangle$  where  $\mathcal{L}$  and  $\mathcal{K}$  define two null eigendirections.

As a consequence of lemma 1, we also have  $t \equiv t_2 = t_3 = \tau_1$ ,  $t_1 = 4t$  and  $\tau_2 = \tau_3 = -2t$ . Then, from (2) and (11), and taking al into account relations given in Appendix B, we can compute the BR tensor and obtain the following:

**Proposition 10** *A BR tensor of type D admits the canonical expression*

$$T = 4t \Pi \otimes \Pi - 2t(\Omega \tilde{\otimes} N + \overline{\Omega} \tilde{\otimes} \overline{N}) + t(\Lambda \tilde{\otimes} K + M \tilde{\otimes} \overline{M}) \quad (12)$$

where the (complex) null frame  $\{\Pi, \Lambda, K, N, \overline{N}, \Omega, \overline{\Omega}, M, \overline{M}\}$  is defined in terms of the Weyl canonical frame  $\{\mathcal{U}, \mathcal{L}, \mathcal{K}\}$  as in (66). The eigenvalue  $4t$  is simple and it is associated with the eigentensor  $\Pi$ . Associated with the quadruple eigenvalue  $t$  we have the eigenspace  $R_t = \{\Lambda, K, M_R, M_I\}$ . Associated with the quadruple eigenvalue  $-2t$  we have the eigenspace  $R_{-2t} = \{N_R, \Omega_R, N_I, \Omega_I\}$ .

### 3.4 Type III

A type III Weyl tensor has a triple vanishing eigenvalue and admits the canonical expression [18]:

$$\mathcal{W} = \mathcal{U} \tilde{\otimes} \mathcal{L} \quad (13)$$



where the *canonical null bivector*  $\mathcal{L}$  is the sole eigenvector that the Weyl tensor admits. Moreover, a null (Jordan) frame of SD bivectors  $\{\mathcal{L}, \mathcal{U}, \mathcal{K}\}$  may be completed.

In this case all the BR eigenvalues vanish as a consequence of lemma 1. Then, from (2) and (13), and taking into account relations given in Appendix B, we can compute the BR tensor and obtain the following:

**Proposition 11** *A BR tensor of type III admits the canonical expression*

$$T = \Lambda \tilde{\otimes} \Pi + N \tilde{\otimes} \bar{N} \quad (14)$$

where the (complex) null frame  $\{\Pi, \Lambda, K, N, \bar{N}, \Omega, \bar{\Omega}, M, \bar{M}\}$  is defined in terms of the Weyl canonical frame  $\{\mathcal{U}, \mathcal{L}, \mathcal{K}\}$  as in (66). Associated with the sole vanishing eigenvalue we have a bidimensional eigenspace  $E = \langle M, \bar{M} \rangle$  and three invariant subspaces:  $I_1 = \langle \Pi, \Lambda, K \rangle$ , containing the sole eigentensor  $\Lambda$ ;  $I_2 = \langle \Omega, \bar{N} \rangle$ , containing the sole eigentensor  $\bar{N}$ ; and  $I_3 = \langle \bar{\Omega}, N \rangle$ , containing the sole eigentensor  $N$ .

### 3.5 Type N

A type N Weyl tensor also has a triple vanishing eigenvalue and it admits the canonical expression [18]:

$$\mathcal{W} = \mathcal{L} \otimes \mathcal{L} \quad (15)$$

Now, the SD bivectors orthogonal to the *canonical null bivector*  $\mathcal{L}$  define a two-dimensional eigenspace.

In this case all the BR eigenvalues also vanish as a consequence of lemma 1. Then, from (2) and (13), and taking into account relations given in Appendix B, we can compute the BR tensor and obtain the following:

**Proposition 12** *A BR tensor of type N admits the canonical expression*

$$T = \Lambda \otimes \Lambda \quad (16)$$

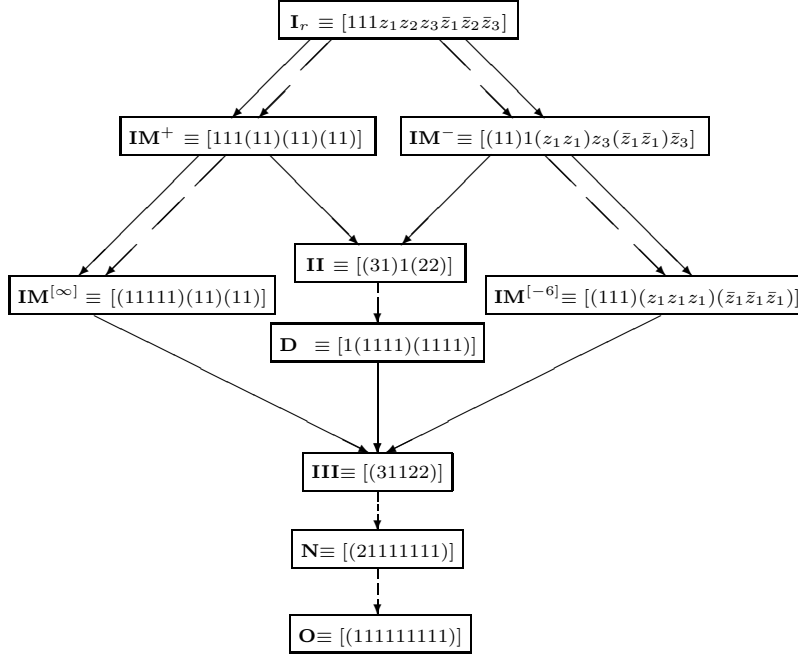
where  $\Lambda = \mathcal{L} \cdot \bar{\mathcal{L}}$ ,  $\mathcal{L}$  being the canonical null bivector. Associated with the sole vanishing eigenvalue we have the eight-dimensional eigenspace orthogonal to  $\Lambda$ .

## 4 Segrè types of the Bel-Robinson tensor

Propositions 1 and 2 summarize the eigenvalue multiplicity and the degree of the minimal polynomial of the different BR classes. Although these results restrict the Segrè type, they do not fix them for the BR classes II, III and N. The study of the eigentensors given in section above completes all the information on these Segrè types.

We present the Segrè types of the BR classes as a flow chart that helps us to visualize the different kinds of degeneration. The continuous arrows correspond to a degeneration in the eigenvalue multiplicity, the four levels

having, respectively, nine (type  $I_r$ ), six (types  $IM^+$  and  $IM^-$ ), three (types  $I^\infty$ ,  $IM^{[-6]}$ , II and D) or one (types III, N and O) different eigenvalues. The dash arrows correspond to a degeneration in the degree of the minimal polynomial: nine (type  $I_r$ ), six (types  $IM^+$ ,  $IM^-$  and II), three (types  $I^\infty$ ,  $IM^{[-6]}$ , D and III), two (type N) or one (type O).



## 5 The Weyl tensor in terms of the Bel-Robinson tensor

The BR tensor can be obtained from the Weyl tensor by means of the quadratic expression (1). This expression is known to be invariant under duality rotation of the Weyl tensor. Thus, we can pose the following question: can the Weyl tensor be determined, up to a duality rotation, from the BR tensor? or, to be more precise, is there an explicit algorithm to obtain it?

In [11] we have solved this problem for the algebraic types where neither  $a$  nor  $b$  vanish. In this 'generic' case we give the Weyl tensor as an explicit concomitant of the BR tensor.

Here we deal with the cases when  $a$  or  $b$  vanish. This condition leads to the algebraic types N, III,  $\text{IM}^\infty$  and  $\text{IM}^{-6}$ , which we study separately. In these 'non generic' cases our approach is based on three steps. First, we express the BR eigentensors in terms of the BR tensor; secondly we obtain the Weyl canonical bivectors from the BR eigentensors; and, finally, we use the Weyl canonical form to get the Weyl tensor.

In the second step, we will use the following three lemmas that may be easily shown from the expressions given in Appendix B:

**Lemma 2** Let us consider the symmetric tensor  $\Lambda = \mathcal{L} \cdot \overline{\mathcal{L}}$ , where  $\mathcal{L}$  is a null SD bivector. Then,  $\Lambda$  determines  $\mathcal{L}$  up to a phase  $\phi$  as

$$\begin{aligned} \mathcal{L} &= e^{i\phi} \mathcal{L}_0, \quad \mathcal{L}_0 \equiv \frac{1}{\sqrt{2}}(L - i * L), \quad L \equiv l \wedge p, \\ l &\equiv -\frac{\Lambda(u)}{\sqrt{\Lambda(u,u)}}, \quad p \equiv \frac{x}{\sqrt{x^2}}, \quad x \equiv *(l \wedge u \wedge w), \end{aligned} \quad (17)$$

where  $u$  is an arbitrary time-like vector and  $w$  an arbitrary vector such that  $x \neq 0$ .

**Lemma 3** Let us consider the symmetric tensors  $\Lambda = \mathcal{L} \cdot \overline{\mathcal{L}}$  and  $\Pi = \mathcal{U} \cdot \overline{\mathcal{U}}$ , where  $\mathcal{L}$  and  $\mathcal{U}$  are two orthogonal SD bivectors, null and unitary, respectively. Then,  $\Lambda$  and  $\Pi$  determine  $\mathcal{L}$  (up to a phase  $\phi$ ) and  $\mathcal{U}$  as

$$\begin{aligned} \mathcal{L} &= e^{i\phi} \mathcal{L}_0, \quad \mathcal{L}_0 \equiv \frac{1}{\sqrt{2}}(L - i * L), \quad L \equiv l \wedge p, \\ l &\equiv -\frac{\Lambda(u)}{\sqrt{\Lambda(u,u)}}, \quad p \equiv \frac{h(w)}{\sqrt{h(w,w)}}, \quad h \equiv \frac{1}{2}g - \Pi, \end{aligned} \quad (18)$$

$$\mathcal{U} = \frac{1}{\sqrt{2}}(U - i * U), \quad U \equiv l \wedge e, \quad e \equiv -\frac{v(u)}{v(l,u)}, \quad v \equiv \frac{1}{2}g + \Pi, \quad (19)$$

where  $u$  is an arbitrary time-like vector and  $w$  an arbitrary vector such that  $h(w) \neq 0$ .

**Lemma 4** Let us consider the symmetric tensor  $\Pi = \mathcal{U} \cdot \overline{\mathcal{U}}$ , where  $\mathcal{U}$  is a unitary SD bivector. Then,  $\Pi$  determine the tensorial square of  $\mathcal{U}$  as

$$\mathcal{U} \otimes \mathcal{U} = \frac{1}{2}(D - i * D), \quad D \equiv -[\Pi \wedge \Pi + \frac{1}{4}g \wedge g]. \quad (20)$$

We summarize the results of the 'generic' case in the next subsection. Afterwards, in the following subsections, we study the four non generic cases separately.

### 5.1 Generic case

The condition  $a \neq 0 \neq b$  can be stated in terms of the BR tensor as  $\text{Tr } T^2 \neq 0 \neq \text{tr } T^3$ . In [11] we have shown the following

**Theorem 1** If  $T$  is a BR tensor and  $\text{Tr } T^2 \neq 0$ ,  $\text{Tr } T^3 \neq 0$ , then the Weyl tensor can be obtained, up to duality rotation, as

$$\mathcal{W} = e^{i\theta} \mathcal{W}_0, \quad \mathcal{W}_0 = \frac{1}{\sqrt{\text{Tr } T^2 \text{tr } T^3}} \left[ 4 \mathcal{T}_{(3)} + \frac{1}{3} \text{Tr } T^3 \mathcal{G} \right] \quad (21)$$

$$\mathcal{T}_{(3)\alpha\beta\rho\sigma} \equiv (T^3)_{\alpha\mu\nu\beta} \mathcal{G}^{\mu\nu}_{\rho\sigma} \quad (22)$$

## 5.2 Type N

A BR tensor of type N is characterized by the condition  $T^2 = 0$  [11]. As stated in subsection 3.5, in this case the Weyl tensor and the BR tensor take, respectively, the expressions (15) and (16), where  $\Lambda = \mathcal{L} \cdot \bar{\mathcal{L}}$ . Then, in a first step, we must obtain  $\Lambda$  in terms of the BR tensor  $T$ . A direct calculation leads to the following result.

**Lemma 5** *If  $T = \Lambda \otimes \Lambda$ , then*

$$\Lambda = \frac{T(X)}{\sqrt{T(X, X)}}, \quad (23)$$

where  $X = u \otimes u$ ,  $u$  being an arbitrary time-like vector.

Secondly, we can obtain  $\mathcal{L}$  (up to a phase  $\phi$ ) in terms of  $\Lambda$  by using lemma 2. The effect of this freedom is a duality rotation on the Weyl tensor obtained as (15). Then, we have:

**Theorem 2** *If  $T$  is a BR tensor such that  $T^2 = 0$ , then the Weyl tensor can be obtained, up to duality rotation, as*

$$\mathcal{W} = e^{i\theta} \mathcal{W}_0, \quad \mathcal{W}_0 = \mathcal{L}_0 \otimes \mathcal{L}_0, \quad (24)$$

where  $\mathcal{L}_0$  can be obtained as (17) in terms of the BR concomitant  $\Lambda$  given in (23).

## 5.3 Type III

A BR tensor of type III is characterized by the conditions  $T^3 = 0$ ,  $T^2 \neq 0$  [11]. As stated in subsection 3.4, in this case the Weyl tensor and the BR tensor take, respectively, the expressions (13) and (14), where  $\Lambda = \mathcal{L} \cdot \bar{\mathcal{L}}$  and  $\Pi = \mathcal{U} \cdot \bar{\mathcal{U}}$ . Then, in a first step, we must obtain  $\Lambda$  and  $\Pi$  in terms of the BR tensor  $T$ .

From (14) we can compute the square  $T^2$  and obtain:

$$T^2 = \Lambda \otimes \Lambda \quad (25)$$

On the other hand, taking into account expressions given in Appendix B, we can also compute the tensor

$$S_{\alpha\beta\mu\nu\rho\sigma} \equiv (T \cdot T)_{\alpha\beta\rho\sigma\mu\nu} + (T \cdot T)_{\alpha\beta\sigma\rho\mu\nu}, \quad (26)$$

and we obtain:

$$S = (\mathbf{N} \tilde{\otimes} \bar{\mathbf{N}}) \otimes \Lambda + (\Lambda \tilde{\otimes} \Pi) \otimes \Lambda + \frac{1}{2} \Lambda \otimes \Lambda \otimes g \quad (27)$$

From (14), (25) and (27) we have:

$$S_{\alpha\beta\mu\nu\rho\sigma} - T_{\alpha\beta\mu\nu} \Lambda_{\rho\sigma} - \frac{1}{2} (T^2)_{\alpha\beta\mu\nu} g_{\rho\sigma} = 2R_{\alpha\beta\mu\nu\rho\sigma} \quad (28)$$

where  $R$  takes the expression:

$$R_{\alpha\beta\mu\nu\rho\sigma} = \Pi_{\alpha\beta}(T^2)_{\mu\nu\rho\sigma} + (T^2)_{\alpha\beta\rho\sigma}\Pi_{\mu\nu} \quad (29)$$

Now, from (25) and (29) we can obtain  $\Lambda$  and  $\Pi$  in terms of the BR concomitants  $T^2$  and  $R$  and we arrive to the following:

**Lemma 6** *If  $T = \Lambda \otimes \Pi + N \otimes \tilde{N}$ , then*

$$\begin{aligned} \Lambda &= \frac{T^2(X)}{\sqrt{T^2(X, X)}}, \\ \Pi &= \frac{1}{T^2(X, X)} \left[ R(X, X) - \frac{R(X, X, X)}{2T^2(X, X)} T^2(X) \right], \end{aligned} \quad (30)$$

where  $X = u \otimes u$ ,  $u$  being an arbitrary time-like vector, and  $R$  is given by:

$$2R_{\alpha\beta\mu\nu\rho\sigma} \equiv (T \cdot T)_{\alpha\beta\rho\sigma\mu\nu} + (T \cdot T)_{\alpha\beta\sigma\rho\mu\nu} - T_{\alpha\beta\mu\nu} \Lambda_{\rho\sigma} - \frac{1}{2}(T^2)_{\alpha\beta\mu\nu} g_{\rho\sigma}. \quad (31)$$

In a second step, we can obtain  $\mathcal{L}$  (up to a phase  $\phi$ ) and  $\mathcal{U}$  in terms of  $\Lambda$  and  $\Pi$  by using lemma 3. The effect of the freedom  $\phi$  is a duality rotation on the Weyl tensor obtained as (13). Then, we have:

**Theorem 3** *If  $T$  is a BR tensor such that  $T^3 = 0$  and  $T^2 \neq 0$ , then the Weyl tensor can be obtained, up to duality rotation, as*

$$\mathcal{W} = e^{i\theta} \mathcal{W}_0, \quad \mathcal{W}_0 = \mathcal{U} \otimes \tilde{\mathcal{L}}_0, \quad (32)$$

where  $\mathcal{L}_0$  and  $\mathcal{U}$  can be obtained as (18) and (19) in terms of the BR concomitants  $\Lambda$  and  $\Pi$  given in (30-31).

#### 5.4 Type $\text{IM}^\infty$

A BR tensor of type  $\text{IM}^\infty$  is characterized by the conditions  $\text{Tr } T^3 = 0$ ,  $\text{Tr } T^2 \neq 0$  [11]. Then, the Weyl tensor has a vanishing eigenvalue and the canonical form (3) becomes:

$$\mathcal{W} = \rho (\mathcal{U}_2 \otimes \mathcal{U}_2 - \mathcal{U}_3 \otimes \mathcal{U}_3). \quad (33)$$

On the other hand, as stated in proposition 6, the BR tensor takes the expression (6), where  $\Pi_i = \mathcal{U}_i \cdot \bar{\mathcal{U}}_i$  and  $\Pi_{jk} = \mathcal{U}_j \cdot \bar{\mathcal{U}}_k (= \bar{\Pi}_{kj})$ ,  $j \neq k$ .

In a first step, we must obtain some concomitants of the eigentensors  $\Pi_i$  in terms of the BR tensor  $T$ . From (6), we can compute  $T^2$  and then obtain

$$P \equiv \frac{1}{t} T + \frac{2}{t^2} T^2 = \Pi_2 \otimes \Pi_2 + \Pi_3 \otimes \Pi_3 \quad (34)$$

Using expressions given in Appendix B we can compute the tensor:

$$V_{\alpha\beta\mu\nu} \equiv 2 P_{\alpha\rho\mu\sigma} P_{\beta}^{\rho}{}_{\nu}{}^{\sigma} - \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} \quad (35)$$

and we arrive to the projector on the eigendirection  $\Pi_1$ :

$$V = \Pi_1 \otimes \Pi_1. \quad (36)$$

Then, we can compute:

$$Q \equiv \Pi_1 \cdot P = \Pi_3 \otimes \Pi_2 + \Pi_2 \otimes \Pi_3 \quad (37)$$

Now, from (34) and (37) we can apply Lemma 9 in Appendix C.1 and arrive to the following:

**Lemma 7** *For a BR tensor  $T$  of type  $IM^\infty$ , let  $\{\Pi_i\}$  be its three first normalized eigentensors. Then we have:*

$$\Pi_1 = \frac{V(X)}{\sqrt{V(X, X)}}, \quad (38)$$

$$\Pi_2 \otimes \Pi_2 - \Pi_3 \otimes \Pi_3 = \frac{P(X) \otimes P(X) - Q(X) \otimes Q(X)}{\sqrt{[P(X, X)]^2 - [Q(X, X)]^2}}, \quad (39)$$

where  $X = u \otimes u$ ,  $u$  being an arbitrary time-like vector such that  $\Pi_1(u) \neq u$ , and

$$P \equiv \frac{1}{t} T + \frac{2}{t^2} T^2, \quad Q \equiv \Pi_1 \cdot P, \quad V_{\alpha\beta\mu\nu} \equiv 2 P_{\alpha\rho\mu\sigma} P_{\beta}^{\rho}{}_{\nu}{}^{\sigma} - \frac{1}{4} g_{\alpha\beta} g_{\mu\nu}. \quad (40)$$

On the other hand, by using lemma 4 we obtain:

$$\mathcal{U}_2 \otimes \mathcal{U}_2 - \mathcal{U}_3 \otimes \mathcal{U}_3 = \frac{1}{2}(D - i * D), \quad D \equiv \Pi_3 \wedge \Pi_3 - \Pi_2 \wedge \Pi_2. \quad (41)$$

Moreover, from lemma 1 we have  $\rho = e^{i\theta} \sqrt{t}$ , where  $\theta$  is an arbitrary phase that gives a duality rotation in the expression (33) of the Weyl tensor. Then, taking into account lemma 7 and expression (41) we have:

**Theorem 4** *If  $T$  is a BR tensor such that  $\text{Tr } T^3 = 0$  and  $4t^2 \equiv \text{Tr } T^2 \neq 0$ , then the Weyl tensor can be obtained, up to duality rotation, as*

$$W = \cos \theta W_0 + \sin \theta * W_0, \quad W_0 = \sqrt{t} \frac{P(X) \wedge P(X) - Q(X) \wedge Q(X)}{\sqrt{[P(X, X)]^2 - [Q(X, X)]^2}}, \quad (42)$$

where  $X = u \otimes u$ ,  $u$  being an arbitrary time-like vector such that  $\Pi_1(u) \neq u$ , and where  $\Pi_1$  is given in (38) and  $P$ ,  $Q$  and  $V$  are given in (40).

### 5.5 Type IM<sup>[-6]</sup>

A BR tensor of type IM<sup>[-6]</sup> is characterized by the conditions  $\text{Tr} T^3 \neq 0$ ,  $\text{Tr} T^2 = 0$  [11]. Then, the three Weyl tensor eigenvalues differ in a cubic root of the unity, and the canonical form (3) becomes:

$$\mathcal{W} = \rho (\lambda_1 \mathcal{U}_1 \otimes \mathcal{U}_1 + \lambda_2 \mathcal{U}_2 \otimes \mathcal{U}_2 + \lambda_3 \mathcal{U}_3 \otimes \mathcal{U}_3) , \quad (43)$$

where  $\lambda_i^3 = 1$ . On the other hand, as stated in proposition 8, the BR tensor takes the expression (8). This expression may be written:

$$T = t(A + \bar{\lambda}_1 \lambda_2 L + \lambda_1 \bar{\lambda}_2 \bar{L}) , \quad (44)$$

where  $L = \Pi_{13} \otimes \Pi_{13} + \Pi_{32} \otimes \Pi_{32} + \Pi_{21} \otimes \Pi_{21}$  and

$$A = \Pi_1 \otimes \Pi_1 + \Pi_2 \otimes \Pi_2 + \Pi_3 \otimes \Pi_3 . \quad (45)$$

In a first step, we must obtain some concomitants of the eigentensors  $\Pi_i$  in terms of the BR tensor  $T$ . From (45), and taking into account expressions given in Appendix B, we can compute the powers of  $T$ :

$$T^2 = t^2(A + \lambda_1 \bar{\lambda}_2 L + \bar{\lambda}_1 \lambda_2 \bar{L}) , \quad T^3 = t^3(A + L + \bar{L})$$

From these expressions and (45) we obtain:

$$A = \frac{1}{3} \left[ \frac{1}{t} T + \frac{1}{t^2} T^2 + \frac{1}{t^3} T^3 \right] . \quad (46)$$

Moreover, we can define the quadratic and cubic concomitants of  $A$  given by:

$$B_{\alpha\beta\gamma\delta\lambda\mu} = 2 A_{\alpha\beta\lambda}{}^\epsilon A_{\gamma\delta\mu\epsilon} - \frac{1}{2} A_{\alpha\beta\gamma\delta} g_{\lambda\mu} , \quad (47)$$

$$C_{\alpha\beta\gamma\delta\lambda\mu\nu\rho} = 2 A_{\alpha\beta\nu}{}^\epsilon B_{\gamma\delta\lambda\mu\rho\epsilon} - \frac{1}{2} B_{\alpha\beta\gamma\delta\lambda\mu} g_{\nu\rho} , \quad (48)$$

and, if we compute them taking into account expressions given in Appendix B, we obtain:

$$B = \sum_{\sigma \in \Sigma_3} \Pi_{\sigma(1)} \otimes \Pi_{\sigma(2)} \otimes \Pi_{\sigma(3)} , \quad C = \sum_{i < j} [\Pi_i \tilde{\otimes} \Pi_j] \otimes [\Pi_i \tilde{\otimes} \Pi_j] . \quad (49)$$

Now, from (45) and (49) we can apply Lemma 10 in Appendix C.2 and arrive to the following:

**Lemma 8** *For a BR tensor  $T$  of type IM<sup>[-6]</sup>, let  $\{\Pi_i\}$  be its three first normalized eigentensors. Then the tensor  $H = \lambda_1 \Pi_1 \otimes \Pi_1 + \lambda_2 \Pi_2 \otimes \Pi_2 + \lambda_3 \Pi_3 \otimes \Pi_3$ , where  $\lambda_i$  are the three cubic roots of the unity, can be obtained in terms of the BR concomitants  $A$ ,  $B$  and  $C$  given in (46), (47) and (48) as:*

$$H = \frac{1}{2w\Delta} [-(\alpha w^3 + 2r\alpha^2 + r^2)A + (w^3 + r\alpha)E + rI] \quad (50)$$



where

$$E \equiv A(X) \otimes A(X) - \frac{1}{2}C(X, X), \quad I \equiv \frac{1}{4}B(X, X) \otimes B(X, X) - \beta B(X), \quad (51)$$

$$\alpha \equiv \frac{1}{3}A(X, X), \quad \beta \equiv \frac{1}{6}B(X, X, X) \quad \gamma \equiv \frac{1}{12}C(X, X, X, X), \quad (52)$$

$$\Delta \equiv \sqrt{s^2 + r^3}, \quad r \equiv \gamma - \alpha^2, \quad s \equiv \frac{1}{2}(3\alpha\gamma - \beta^2) - \alpha^3, \quad (53)$$

and where  $w$  is a nonvanishing scalar defined by one of the expressions  $w^3 \equiv -s \pm \Delta$ , and  $X$  being an arbitrary symmetric tensor such that  $\Delta \neq 0$ .

On the other hand, by using lemma 4 we obtain:

$$\begin{aligned} \lambda_1 \mathcal{U}_1 \otimes \mathcal{U}_1 + \lambda_2 \mathcal{U}_2 \otimes \mathcal{U}_2 + \lambda_3 \mathcal{U}_3 \otimes \mathcal{U}_3 &= \frac{1}{2}(J - \mathbf{i} * J), \\ J &\equiv -[\lambda_1 \Pi_1 \wedge \Pi_1 + \lambda_2 \Pi_2 \wedge \Pi_2 + \lambda_3 \Pi_3 \wedge \Pi_3]. \end{aligned} \quad (54)$$

Moreover, from lemma 1 we have  $\rho = e^{\mathbf{i}\theta} \sqrt{t}$ , where  $\theta$  is an arbitrary phase that gives a duality rotation in the expression (43) of the Weyl tensor. Then, taking into account lemma 8 and expression (54) we have:

**Theorem 5** *If  $T$  is a BR tensor such that  $\text{Tr } T^2 = 0$  and  $9t^3 \equiv \text{Tr } T^3 \neq 0$ , then the Weyl tensor can be obtained, up to duality rotation, as*

$$W = \cos \theta W_0 + \sin \theta * W_0, \quad W_0 = \sqrt{t} J, \quad J_{\alpha\beta\mu\nu} \equiv 2(H_{\alpha\mu\beta\nu} - H_{\alpha\nu\beta\mu}), \quad (55)$$

where  $H$  is the concomitant of the BR tensor given in expression (50) of lemma 8.

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## A Notation

### A.1 Products and other formulas involving 2-tensors $A$ and $B$

1. Composition as endomorphisms:  $A \cdot B$ ,

$$(A \cdot B)^\alpha{}_\beta = A^\alpha{}_\mu B^\mu{}_\beta$$

2. In general, for arbitrary tensors,  $\cdot$  will be used to indicate the contraction of adjacent indexes on the tensorial product.
3. Square and trace as endomorphism

$$A^2 = A \cdot A, \quad \text{tr } A = A^\alpha{}_\alpha$$

4. Action on vectors  $x, y$  as an endomorphism  $A(x)$  and as a bilinear form  $A(x, y)$ :

$$A(x)^\alpha = A^\alpha{}_\beta x^\beta, \quad A(x, y) = A_{\alpha\beta} x^\alpha y^\beta$$

5. Exterior product as double 1-forms:  $A \wedge B$ ,

$$(A \wedge B)_{\alpha\beta\mu\nu} = A_{\alpha\mu} B_{\beta\nu} + B_{\alpha\mu} A_{\beta\nu} - A_{\alpha\nu} B_{\beta\mu} - B_{\alpha\nu} A_{\beta\mu}$$

### A.2 Products and other formulas involving SD-endomorphisms $\mathcal{X}$ and $\mathcal{Y}$

1. Every self-dual (SD) symmetric double 2-form  $\mathcal{X}$  defines a linear map on the 3-dimensional SD bivector space. In short, we will say that  $\mathcal{X}$  is a SD-endomorphism.
2. Composition as endomorphisms  $\mathcal{X} \circ \mathcal{Y}$ :

$$(\mathcal{X} \circ \mathcal{Y})_{\alpha\beta\rho\sigma} = \frac{1}{2} \mathcal{X}^{\alpha\beta}{}_{\mu\nu} \mathcal{Y}^{\mu\nu}{}_{\rho\sigma}$$

3. Square and trace as endomorphism:

$$\mathcal{X}^2 = \mathcal{X} \circ \mathcal{X}, \quad \text{Tr } \mathcal{X} = \frac{1}{2} \mathcal{X}^{\alpha\beta}{}_{\alpha\beta}$$

4. Action (on SD bivectors  $\mathcal{F}$ ,  $\mathcal{H}$ ) as an endomorphism  $\mathcal{X}(\mathcal{F})$ , and as a bilinear form  $\mathcal{X}(\mathcal{F}, \mathcal{H})$ :

$$\mathcal{X}(\mathcal{F})_{\alpha\beta} = \frac{1}{2} \mathcal{X}_{\alpha\beta}{}^{\mu\nu} \mathcal{F}_{\mu\nu}, \quad \mathcal{X}(\mathcal{F}, \mathcal{H}) = \frac{1}{4} \mathcal{X}_{\alpha\beta\mu\nu} \mathcal{F}^{\alpha\beta} \mathcal{H}^{\mu\nu}$$

5. Metric on the space of SD bivectors (SD-identity):

$$\mathcal{G} = \frac{1}{2} (G - \text{i } \eta), \quad \mathcal{G}(\mathcal{F}) = \mathcal{F}$$

6. The metric volume element  $\eta$  is a linear map on the 2-forms space that defines the Hodge dual operator. For a real 2-form  $F$  and a real symmetric double 2-form  $W$ :

$$*F = \eta(F), \quad *W = \eta \circ W.$$

### A.3 Products and other formulas involving TLS-endomorphisms $T$ and $R$ .

1. Every 4-tensor  $T$  with the symmetries:

$$T_{\alpha\beta\mu\nu} = T_{\beta\alpha\mu\nu} = T_{\mu\nu\alpha\beta}, \quad T^{\alpha}{}_{\alpha\mu\nu} = 0 \quad (56)$$

defines a symmetric linear map on the 9-dimensional space of the traceless symmetric tensors. We say that  $T$  is a TLS-endomorphism.

2. Composition as endomorphisms:  $T \bullet R$ ,

$$(T \bullet R)^{\alpha\beta}{}_{\rho\sigma} = T^{\alpha\beta}{}_{\mu\nu} R^{\mu\nu}{}_{\rho\sigma}$$

3. Square and trace as endomorphism:

$$T^2 = T \bullet T, \quad \text{Tr } T = T^{\alpha\beta}{}_{\alpha\beta}$$

4. Action (on traceless symmetric tensors  $A$ ,  $B$ ) as an endomorphism  $T(A)$  and as a bilinear form  $T(A, B)$ ,

$$T(A)_{\alpha\beta} = T_{\alpha\beta}{}^{\mu\nu} A_{\mu\nu}, \quad T(A, B) = T_{\alpha\beta\mu\nu} A^{\alpha\beta} B^{\mu\nu}$$

5. Metric on the space of traceless symmetric tensors (TLS-identity):  $\Gamma$

$$\Gamma_{\alpha\beta\mu\nu} = \frac{1}{2} (g_{\alpha\mu} g_{\beta\nu} + g_{\alpha\nu} g_{\beta\mu}) - \frac{1}{4} g_{\alpha\beta} g_{\mu\nu}, \quad \Gamma(A) = A$$

## B Frames

### B.1 Frames of vectors

1. In a spacetime with signature  $(-, +, +, +)$  we shall note  $\{e_\alpha\}$  an orthonormal frame of vector fields having the orientation  $\eta = e_0 \wedge e_1 \wedge e_2 \wedge e_3$ .
2. A (complex) null frame  $\{l, k, m, \bar{m}\}$  is defined from the orthonormal frame by

$$l = \frac{1}{\sqrt{2}}(e_0 - e_1), \quad k = \frac{1}{\sqrt{2}}(e_0 + e_1), \quad m = \frac{1}{\sqrt{2}}(e_2 + i e_3) \quad (57)$$

### B.2 Frames of SD bivectors

1. We shall note  $\{\mathcal{U}_i\}$  the orthonormal frame of SD bivectors defined by

$$\mathcal{U}_i = \frac{1}{\sqrt{2}}(U_i - i * U_i), \quad U_i = e_0 \wedge e_i \quad (58)$$

2. The frame  $\{\mathcal{U}_i\}$  satisfies

$$\mathcal{U}_i \cdot \mathcal{U}_j = \frac{1}{2} \delta_{ij} g - \frac{i}{\sqrt{2}} \epsilon_{ijk} \mathcal{U}_k, \quad \epsilon_{123} = 1 \quad (59)$$

3. The metric  $\mathcal{G} = \frac{1}{2}(G - i \eta)$ , in the space of SD bivectors is  $\mathcal{G} = -\sum \mathcal{U}_i \otimes \mathcal{U}_i$ .
4. Associated with a null frame  $\{l, k, m, \bar{m}\}$  we have the SD null frame  $\{\mathcal{U}, \mathcal{L}, \mathcal{K}\}$ , defined by the relations

$$\mathcal{U} = \frac{1}{\sqrt{2}}(l \wedge k - m \wedge \bar{m}), \quad \mathcal{L} = l \wedge m, \quad \mathcal{K} = k \wedge \bar{m} \quad (60)$$

5. The metric  $\mathcal{G}$  is given by  $\mathcal{G} = -\mathcal{U} \otimes \mathcal{U} - \mathcal{L} \otimes \tilde{\mathcal{K}}$ .
6. The SD frames  $\{\mathcal{U}_i\}$  and  $\{\mathcal{U}, \mathcal{L}, \mathcal{K}\}$  are related by

$$\mathcal{U} = \mathcal{U}_1, \quad \mathcal{L} = \frac{1}{\sqrt{2}}(\mathcal{U}_2 + i \mathcal{U}_3), \quad \mathcal{K} = \frac{1}{\sqrt{2}}(\mathcal{U}_2 - i \mathcal{U}_3) \quad (61)$$

### B.3 Frames of symmetric tensors

1. From every orthonormal frame  $\{\mathcal{U}_i\}$  of SD bivectors we can define the (complex) orthogonal frame of traceless symmetric tensors  $\{\Pi_{ij}\}$  defined as

$$\Pi_{ij} = \mathcal{U}_i \cdot \bar{\mathcal{U}}_j \quad (62)$$

2. The frame  $\{\Pi_{ij}\}$  satisfies

$$(\Pi_{ij}, \Pi_{km}) = \delta_{ik} \delta_{jm} \quad (63)$$

3. In terms of the frame  $\{e_\alpha\}$  the frame  $\{\Pi_{ij}\}$  takes the expression:

$$\Pi_i \equiv \Pi_{ii} = \frac{1}{2}(v_i - h_i), \quad \Pi_{ij} = \frac{1}{2}(e_i \otimes e_j + i \epsilon_{ijk} e_0 \otimes e_k) \quad (i \neq j) \quad (64)$$

where  $v_i = -e_0 \otimes e_0 + e_i \otimes e_i$ , and  $h_i = g - v_i$ .

4. The tern  $\{\Pi_i\}$  satisfies

$$\Pi_i^2 = \frac{1}{4}g; \quad \Pi_i \cdot \Pi_j = \frac{1}{2}\Pi_k, \quad (i, j, k \neq). \quad (65)$$

5. The metric  $\Gamma$  in the space of traceless symmetric tensors is  $\Gamma = \sum_{i,j} \Pi_{ij} \otimes \Pi_{ij}$ .

6. From every null frame  $\{\mathcal{U}, \mathcal{L}, \mathcal{K}\}$  of SD bivectors we can define the (complex) null frame of traceless symmetric tensors  $\{\Pi, \Lambda, K, N, \bar{N}, \bar{\Omega}, \bar{M}, \bar{M}\}$  defined as

$$\begin{aligned} \Pi &= \mathcal{U} \cdot \bar{\mathcal{U}} & \Lambda &= \mathcal{L} \cdot \bar{\mathcal{L}} & K &= \mathcal{K} \cdot \bar{\mathcal{K}} \\ N &= \mathcal{U} \cdot \bar{\mathcal{L}} & \bar{\Omega} &= \mathcal{U} \cdot \bar{\mathcal{K}} & M &= \mathcal{L} \cdot \bar{\mathcal{K}} \end{aligned} \quad (66)$$

7. In terms of the frame  $\{l, k, m, \bar{m}\}$  the frame  $\{\Pi, \Lambda, K, N, \bar{N}, \bar{\Omega}, \bar{M}, \bar{M}\}$  takes the expression:

$$\begin{aligned} \Pi &= -\frac{1}{2}(l \otimes k + m \otimes \bar{m}), & N &= -\frac{1}{\sqrt{2}}l \otimes \bar{m}, & \bar{\Omega} &= \frac{1}{\sqrt{2}}k \otimes m, \\ M &= m \otimes m, & \Lambda &= -l \otimes l, & K &= -k \otimes k \end{aligned} \quad (67)$$

8. The metric  $\Gamma$  in the space of traceless symmetric tensors is

$$\Gamma = \Pi \otimes \Pi + \Lambda \otimes \bar{\Lambda} + M \otimes \bar{M} + N \otimes \bar{N} + \bar{\Omega} \otimes \bar{\Omega} \quad (68)$$

## C Some technical results

C.1.- Let  $\{e_2, e_3\}$  be two orthonormal vectors of an Euclidean vectorial space and let us suppose that the tensors  $P = e_2 \otimes e_2 + e_3 \otimes e_3$  and  $Q = e_2 \otimes e_3$  are known. Firstly, we want to obtain  $\{e_2, e_3\}$  in terms of  $Q$  and  $V$ . Let us define

$$\kappa = Q(x, x), \quad \nu = V(x, x)$$

where  $x$  is whatever vector such that  $\kappa^2 - \nu^2 \neq 0$ . Let us take  $\lambda_1$  one of the roots of the quadratic equation

$$\lambda^2 - \kappa\lambda + \frac{1}{4}\nu^2 = 0$$

and let us define  $p = \sqrt{\lambda_1}$ ,  $q = \frac{\nu}{p}$ . Then  $e_2$  and  $e_3$  are given by

$$e_2 = \frac{1}{p^2 - q^2}[pP(x) - qQ(x)], \quad e_3 = \frac{1}{p^2 - q^2}[pQ(x) - qP(x)] \quad (69)$$

Note that the pair  $\{e_2, e_3\}$  is determined up to ordination and sign, accordingly with the choice of the root of the quadratic equation and the sign in the square root of  $\lambda_1$ .

From expressions (69), a straightforward calculation leads to:

**Lemma 9** *Let  $\{e_2, e_3\}$  be two orthonormal vectors of an Euclidean vectorial space.*

*If  $P = e_2 \otimes e_2 + e_3 \otimes e_3$  and  $Q = e_2 \otimes e_3$ , then:*

$$e_2 \otimes e_2 - e_3 \otimes e_3 = \frac{P(x) \otimes P(x) - Q(x) \otimes Q(x)}{\sqrt{[P(x, x)]^2 - [Q(x, x)]^2}}$$

*where  $x$  is an arbitrary vector such that  $[P(x, x)]^2 \neq [Q(x, x)]^2$ .*

C.2.- Let  $\{e_1, e_2, e_3\}$  be three orthonormal vectors of an Euclidean vectorial space and let us suppose that one knows the tensors

$$A = \sum_{i=1}^3 e_i \otimes e_i, \quad B = \sum_{\sigma \in \Sigma_3} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}, \quad C = \sum_{i < j} [e_i \tilde{\otimes} e_j] \otimes [e_i \tilde{\otimes} e_j]. \quad (70)$$

Let us define

$$\alpha = \frac{1}{3}A(x, x), \quad \beta = \frac{1}{6}B(x, x, x) \quad \gamma = \frac{1}{12}C(x, x, x, x) \quad (71)$$

where  $x$  is whatever vector such  $\Delta \neq 0$  where

$$\Delta \equiv \sqrt{r^2 + s^3}, \quad r \equiv \gamma - \alpha^2, \quad s \equiv \frac{1}{2}(3\alpha\gamma - \beta^2) - \alpha^3. \quad (72)$$

Then, the tensors

$$E = A(x) \otimes A(x) - \frac{1}{2}C(x, x), \quad I = \frac{1}{4}B(x, x) \otimes B(x, x) - \beta B(x), \quad (73)$$

take, in terms of  $e_i$ , the expressions:

$$E = \sum_{i=1}^3 y_i e_i \otimes e_i, \quad I = \sum_{(ijk)} y_i y_j e_k \otimes e_k, \quad (74)$$

for  $(i, j, k)$  a pair permutation of  $(1, 2, 3)$ , and where  $y_i$  are the solutions of the cubic equation

$$y^3 - 3\alpha y^2 + 3\gamma y - \beta^2 = 0$$

Then, if one writes the solutions  $y_i$  in terms of the cubic roots of the unity, a straightforward calculation leads to:

**Lemma 10** *Let  $\{e_1, e_2, e_3\}$  be three orthonormal vectors of an Euclidean vectorial space. Let  $A$ ,  $B$  and  $C$  be the tensors given in (70) and let  $\lambda_i$  be the cubic roots of the unity. Then:*

$$\sum_{i=1}^3 \lambda_i e_i \otimes e_i = \frac{1}{2w\Delta} [-(\alpha w^3 + 2r\alpha^2 + r^2)A + (w^3 + r\alpha)E + rI]$$

where the tensors  $E$  and  $I$  are given in (73) and the scalars  $\alpha$ ,  $\beta$  and  $\gamma$  are defined in (71),  $x$  being an arbitrary vector such that  $\Delta \neq 0$ , and where  $r$ ,  $s$  and  $\Delta$  are defined in (72), and  $w$  is a nonvanishing scalar defined by one of the expressions  $w^3 \equiv -s \pm \Delta$ .

Note that the expression obtained in lemma above is determined up to ordination of the three  $\lambda_i$ , accordingly with the choice of the root in expression of  $w^3$ . In this expression the sign before  $\Delta$  must be chosen such that  $w \neq 0$ .

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